

N=4 supersymmetric integrable systems

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dedicated to the memory of Dmitriy Volkov
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I give an overview of recent progress in constructing the KdV, mKdV and NLS type hierarchies with extended $N = 4$ supersymmetry.

1. Introduction. It is widely believed nowadays that the ultimate theory of all fundamental forces is by no means a standard field theory, but rather that of extended objects like superstrings and supermembranes. Supersymmetry, the concept pioneered by D.V. Volkov [1], will surely be one of the key-stones of this future theory. Affine and W algebras and superalgebras are also expected to be necessary ingredients of the underlying symmetry structure of this theory as they naturally come out as the world-sheet (or world-volume) gauge symmetries of extended objects.

The WZW, Liouville - Toda and KdV type $2D$ integrable systems are intimately related to these symmetries. They are encountered and proved to be of high relevancy in a plenty of problems of modern mathematical physics (see, e.g., [2]): in non-perturbative $2D$ gravity and related matrix models, in the geometric approaches to strings, superstrings and supermembranes, in Seiberg-Witten non-perturbative approach to supersymmetric Yang-Mills theory, etc. KdV, mKdV, NLS and KP type hierarchies of evolution equations exhibit a remarkable relationship with conformal, affine and W algebras: the latter provide a hamiltonian structure for the former [3]. Supersymmetric extensions of these hierarchies and their interplay with superaffine and superconformal (W) algebras were under intense study for the last decade [4-10].

The subject of my talk is the KdV type integrable hierarchies with extended $N = 4$ supersymmetry. While a lot was known about $N = 1$ and $N = 2$ extensions, until recently it remained unclear whether consistent higher N hierarchies of this kind exist. Only $N = 4$ extensions of some exactly solvable Lorentz-covariant $2D$ systems, Liouville and WZW models, were known [11, 12]. Seeking higher N hierarchies is of extreme interest, in particular, because such systems can be relevant to the program of “grand-unification” of all known hierarchies: apparently unrelated lower supersymmetry (and purely bosonic) integrable systems can turn out to be various reductions of a single higher N supersymmetric system. This phenomenon can be already seen while passing from $N = 1$ KdV hierarchy [6, 7] to the $N = 2$ ones [8, 9]. The latter naturally incorporate both KdV and mKdV hierarchies in the bosonic sector.

In a series of papers [13-15] the first example of KdV system with higher supersymmetry, $N = 4$ KdV hierarchy, was constructed and analyzed, both in a manifestly supersymmetric $N = 4$ superfield form [13] and via $N = 2$ superfields [14]. It was found to be bi-hamiltonian, to have “small” $N = 4$ SCA [16] as the second hamiltonian structure and to possess two different Lax formulations in terms of $N = 2$ super pseudo-differential operators [17, 15]. A remarkable interplay between integrability of this system and breaking of the global automorphism $SU(2)$ symmetry of $N = 4$ supersymmetry was revealed: it is integrable only provided this $SU(2)$ is explicitly broken and the square of $SU(2)$ breaking parameter is proportional to the inverse of the central charge of $N = 4$ SCA. It encompasses two different $N = 2$ KdV hierarchies, the $a = 4$ and $a = -2$ ones, as its two non-equivalent consistent reductions. Later it was found that “small” $N = 4$ SCA provides a hamiltonian structure for one more integrable hierarchy which is an extension of the $a = -2$, $N = 2$ KdV hierarchy and possesses only $N = 2$ global supersymmetry [18]. Recently [19], a simplest $N = 4$ supersymmetric affine hierarchy was constructed. It is defined on $N = 2$ extension of the bosonic affine algebra $\widehat{sl(2) \oplus u(1)}$ and underlies $N = 4$ KdV hierarchy much like ordinary mKdV hierarchy underlies (via a Miura map) the KdV hierarchy. It seems that any $N = 2$ affine algebra or superalgebra admitting a quaternionic structure exhibits hidden $N = 4$ supersymmetry and hence can give rise to $N = 4$ supersymmetric hierarchies. This provides a general clue to constructing and classifying such hierarchies.

2. KdV example. To explain the basic idea of how to construct KdV type hierarchies via relating them to some infinite-dimensional algebras as second hamiltonian structure, let us start with the text-book KdV example.

As was shown in [3], the KdV equation

$$\dot{u} = -u''' + 6uu' \quad (1)$$

can be treated as a hamiltonian system,

$$\dot{u} = \{u, \mathcal{H}_3\} \quad ,$$

with the hamiltonian and the Poisson brackets defined by

$$\mathcal{H}_3 = \frac{1}{2} \int dx \, u^2(x) \quad , \quad \{u(x), u(y)\} = [-\partial^3 + 4u\partial + 2u'] \delta(x - y) \quad . \quad (2)$$

For the Fourier modes of $u(x)$,

$$u(x) = \frac{6}{c} \sum_n \exp(-inx) L_n - \frac{1}{4} \quad , \quad (3)$$

the Poisson brackets in (2) imply the structure relations of the Virasoro algebra

$$i \{L_n, L_m\} = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m, 0} \quad . \quad (4)$$

So, the definition (2) means that the density of the KdV hamiltonian \mathcal{H}_3 is the square of a conformal stress-tensor $u(x)$ obeying the Virasoro algebra (2), (4). One says that the Virasoro algebra provides the second hamiltonian (or Poisson) structure for the KdV equation (historically, first hamiltonian formulation of KdV hierarchy was based upon a linear Poisson algebra,

and the latter is referred to as the first hamiltonian structure). The higher order conserved quantities of the KdV equation can be regarded as the hamiltonians which generate, through the Poisson brackets (2), next equations from the KdV hierarchy.

3. N=1,2 KdVs. The same idea was applied for constructing $N = 1$ and $N = 2$ superextensions of the KdV equation [5-9]. They were related in an analogous way, via the second hamiltonian structure, to $N = 1$ and $N = 2$ SCAs. In the $N = 1$ case the basic object is $N = 1$ stress-tensor, the spin 3/2 fermionic $N = 1$ superfield

$$\Phi(t, x, \theta) = \xi(t, x) + \theta u(t, x). \quad (5)$$

It comprises the spin 3/2 fermionic current ξ and spin 2 stress-tensor u which generate, via appropriate PBs, the classical $N = 1$ SCA. The most general $N = 1$ supersymmetric dimension 3 hamiltonian reads

$$\mathcal{H}_3 = \int dx d\theta \Phi D\Phi, \quad D = \frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial x}, \quad D^2 = \partial_x, \quad (6)$$

and $N = 1$ KdV equation is the equation defining evolution of Φ with respect to \mathcal{H}_3

$$\dot{\Phi} = \{\Phi, \mathcal{H}_3\}, \quad (7)$$

with the superfield PB structure amounting to $N = 1$ SCA.

In the $N = 2$ case one deals with the $N = 2$ stress-tensor which is a spin 1 $N = 2$ superfield

$$J(t, x, \theta, \bar{\theta}) = j(t, x) + \theta \xi(t, x) + \bar{\theta} \bar{\xi}(t, x) + \theta \bar{\theta} u(t, x), \quad (8)$$

with the components being currents of $N = 2$ SCA. Once again, the $N = 2$ KdV equation can be defined as an evolution equation

$$\dot{J} = \{J, \mathcal{H}_3\} \quad (9)$$

with respect to the most general $N = 2$ supersymmetric dimension 3 hamiltonian

$$\mathcal{H}_3 = \int dx d\theta d\bar{\theta} \left(J[D, \bar{D}]J + \frac{a}{6} J^3 \right) \quad (10)$$

and the PB structure

$$\{J(1), J(2)\} = \left(J\partial + \partial J + DJ\bar{D} + \bar{D}JD + \partial[D, \bar{D}] \right) \delta(1, 2), \quad (11)$$

which generates $N = 2$ SCA (we always choose the central charge equal to some number since on the classical level it can be fixed at will by proper rescalings of superfields and PBs). In these formulas

$$\{D, D\} = 0, \quad \{D, \bar{D}\} = -\partial_x, \quad \delta(1, 2) = \delta(x_1 - x_2)(\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2), \quad (12)$$

and the differential operator in the r.h.s. of (11) is evaluated at the first point of $N = 2$ superspace (all derivatives are assumed to act freely to the right).

We observe two differences of $N = 2$ case compared to the two previous cases. Firstly, $N = 2$ supersymmetry requires two fields in the bosonic sector, the spin 2 stress-tensor $u(t, x)$

and the spin 1 current $j(t, x)$ which generates the affine $\widehat{u(1)}$ subalgebra of $N = 2$ SCA. The bosonic sector of $N = 2$ KdV equation is a coupled system of KdV and mKdV equations for these fields. Secondly, there is a free parameter a in the hamiltonian and, respectively, in $N = 2$ KdV equation. It was shown [8-10] that this equation is completely integrable, i.e. gives rise to an infinite hierarchy of conserved hamiltonians in involution and possesses a Lax formulation, only for the three special values of a

$$a = -2, 4, 1. \quad (13)$$

These are just the values at which the coupled KdV-mKdV system in the bosonic sector of $N = 2$ KdV is integrable.

4. N=4 KdV hierarchy. A natural generalization of $N = 2$ SCA in the list of Ademollo *et al* [16] is the “small” $N = 4$ SCA. Alongside with the conformal stress-tensor, it contains a triplet of the spin 1 currents of affine algebra $\widehat{su(2)}$ and a complex $su(2)$ doublet of the spin 3/2 fermionic currents. It can be formulated in a manifestly supersymmetric way as a set of $N = 4$ superfield PBs [13, 14]. We will prefer here a $N = 2$ superfield notation in which this SCA is represented by the $N = 2$ stress-tensor J and chiral and anti-chiral spin 1 supercurrents $\Phi, \bar{\Phi}, D\Phi = \bar{D}\bar{\Phi} = 0$ [14]. Together with (11), the PBs

$$\begin{aligned} \{J(1), \Phi(2)\} &= -(\Phi\bar{D}D + \bar{D}\Phi D)\delta(1, 2), \quad \{J(1), \bar{\Phi}(2)\} = -(\bar{\Phi}D\bar{D} + D\bar{\Phi}\bar{D})\delta(1, 2), \\ \{\Phi(1), \bar{\Phi}(2)\} &= (\partial D\bar{D} + D J\bar{D})\delta(1, 2), \quad \{\Phi(1), \Phi(2)\} = 0 \end{aligned} \quad (14)$$

form the classical “small” $N = 4$ SCA.

In terms of these supercurrents the transformations promoting manifest $N = 2$ supersymmetry to $N = 4$ are given by

$$\delta J = -\epsilon\bar{D}\Phi - \bar{\epsilon}D\bar{\Phi}, \quad \delta\Phi = \bar{\epsilon}DJ, \quad \delta\bar{\Phi} = \epsilon\bar{D}J. \quad (15)$$

It is straightforward to check covariance of (11), (14) under these transformations. Then the problem of constructing $N = 4$ KdV is reduced to constructing most general $N = 4$ supersymmetric dimension 3 hamiltonian out of $J, \Phi, \bar{\Phi}$. It is given by the following expression

$$\mathcal{H}_3 = \int dx d\theta d\bar{\theta} \left\{ J[D, \bar{D}]J - 2\Phi'\bar{\Phi} + \frac{a}{6}J^3 - aJ\Phi\bar{\Phi} - \frac{1}{2}b(\Phi^2 + \bar{\Phi}^2) \right\} \quad (16)$$

and contains two real parameters a and b , arbitrary for the moment. The evolution equations for $J, \Phi, \bar{\Phi}$ can be constructed in the standard way, their explicit form can be found in [14]. They also include the parameters a and b .

In ref. [14] we investigated the issue of existence of higher-order non-trivial conserved hamiltonians for this $N = 4$ KdV system and found that they exist only for the following three options

$$(i). \quad a = 4, \quad b = 0; \quad (ii). \quad a = -2, \quad b = 6; \quad (iii). \quad a = -2, \quad b = -6. \quad (17)$$

Just with these choices the $N = 4$ KdV system turns out bi-hamiltonian. Both the existence of non-trivial higher-order conserved quantities and the bi-hamiltonian property were strong indications that $N = 4$ KdV system is integrable and gives rise to the whole hierarchy for these values of the parameters.

These three choices are essentially different only at first sight. Actually, they are related to each other by hidden $SU(2)$ symmetry transformations which form an automorphism group of $N = 4$, $1D$ supersymmetry. The realization of these transformations on $N = 2$ superfields J , Φ , $\bar{\Phi}$ looks not too illuminating; it can be found in ref. [14].

Both $N = 4$ supersymmetry and $SU(2)$ covariance become transparent and manifest while formulating the $N = 4$ KdV system in $N = 4$, $1D$ harmonic superspace [13]. There, the “small” $N = 4$ SCA is represented by the analytic doubly-charged harmonic superfield $V^{++}(\zeta)$ subjected to the supplementary constraint ¹

$$D^{++}V^{++} = 0, \quad \left(D^{++} = u^+{}^i \frac{\partial}{\partial u^-{}^i} + \theta^+ \bar{\theta}^+ \partial_x \right). \quad (18)$$

It restricts the harmonic dependence of V^{++} so that the irreducible set of component fields of the latter amounts to the currents contents of “small” $N = 4$ SCA. In the ordinary $N = 4$ superspace this “harmonic shortness” condition implies

$$V^{++} = V^{(ik)}(x, \theta, \bar{\theta}) u_i^+ u_k^+,$$

while the analyticity is expressed as the following constraints on $V^{(ik)}$

$$D^{(i} V^{kl)} = \bar{D}^{(i} V^{kl)} = 0,$$

D^i, \bar{D}^k being the appropriate spinor derivatives. The automorphism $SU(2)$ symmetry in this manifestly $N = 4$ supersymmetric formulation is realized as rotations of the doublet indices i, k, l .

The $N = 2$ superfields $J, \Phi, \bar{\Phi}$ are first components (up to numerical coefficients) in the decomposition of such V^{12}, V^{11} and V^{22} with respect to the grassmann coordinates which enlarge $N = 2$ superspace to the $N = 4$ one.

The PB structure (11), (14) can be rewritten as a single PB for the superfields V^{++} (it is explicitly given in [13]). The hamiltonian (16), being expressed through V^{++} , takes the following form [13]

$$\mathcal{H}_3 = \int dZ[du] (D^{--}V^{++})^2 + \int d\zeta^{-4}[du] (a^{--})^2 (V^{++})^3. \quad (19)$$

Here, D^{--} is the second harmonic derivative (not preserving the harmonic analyticity), $dZ[du]$ and $d\zeta^{-2}[du]$ are measures of integration over the whole harmonic superspace and its analytic subspace, the $SU(2)$ breaking parameter $a^{--} = a^{(ik)} u_i^- u_k^-$ is needed for balance of the harmonic $U(1)$ charges in the second piece of \mathcal{H}_3 . Now it is a matter of straightforward though tedious calculation to check that the three options (17) just correspond to the three (up to reflections) independent orientations of the $SU(2)$ breaking constant vector a^{ik} ($(a^{ik})^\dagger = -\epsilon_{il}\epsilon_{kt} a^{lt}$)

$$\begin{aligned} (i). \quad a^{12} &= \pm\sqrt{5}, \quad a^{11} = a^{22} = 0; & (ii). \quad a^{12} = 0, \quad a^{11} = a^{22} = \pm i\sqrt{5}; \\ (iii). \quad a^{12} &= 0, \quad a^{11} = -a^{22} = \pm\sqrt{5}, \end{aligned} \quad (20)$$

¹ $\zeta \equiv (x, \theta^+, \bar{\theta}^+, u_i^+, u_k^-)$ are coordinates of an analytic subspace of harmonic $N = 4$, $1D$ superspace[20], $u_i^\pm, u^+{}^i u_i^- = 1$ being harmonic coordinates, $\theta^+, \bar{\theta}^+$ projections of the $N = 4$ grassmann coordinates $\theta^i, \bar{\theta}^k$ on the harmonics u_i^+ .

this vector having in all cases the same fixed norm

$$|a|^2 = -a^{ik}a_{ik} = 2(a^{12}a^{12} - a^{11}a^{22}) = 10. \quad (21)$$

The latter is the main condition for the $N = 4$ KdV system to possess an infinite number of higher-order hamiltonians in involution and to be bi-hamiltonian [13]. If from the beginning we would keep the central charge k of $N = 4$ SCA unfixed, in the r.h.s. of eq. (21) there appeared the factor $\frac{1}{k}$.

In refs. [15, 17] two different $N = 2$ superfield Lax operators for this $N = 4$ KdV hierarchy have been proposed. Both of them are pseudo-differential and are adapted to the first choice in eqs. (17), (20), taking account of the fact that all the three options are indeed equivalent by hidden $SU(2)$ covariance. These Lax operators are given by

$$L_1 = \partial - J - \bar{D}\partial^{-1}(DJ) - F\bar{D}\partial^{-1}(D\bar{F}) + \bar{D}\partial^{-1}(D(F\bar{F})), \quad DF = \bar{D}\bar{F} = 0, \quad (22)$$

$$(\Phi = D\bar{F}, \quad \bar{\Phi} = \bar{D}F)$$

$$L_2 = D\bar{D} + D\bar{D}\partial^{-1}(J + \bar{\Phi}\partial^{-1}\Phi)\partial^{-1}D\bar{D}. \quad (23)$$

In both cases the flows and the corresponding conserved hamiltonians are given by

$$\frac{\partial L}{\partial t_k} = [L_{\geq 1}^k, L], \quad \mathcal{H}_n = \int dx d\theta d\bar{\theta} \text{res} L^n, \quad (24)$$

the suffix ≥ 1 meaning the pure differential part of pseudo-differential operator. Note different definitions of the residue of the pseudo-differential operators: in the first case it is defined as a coefficient before 1, while in the second case as that before $D\bar{D}\partial^{-1}$.

A natural reduction to $N = 2$ KdV systems is to put in (22) - (24)

$$\Phi = \bar{\Phi} = 0, \quad (25)$$

which leads to the $a = 4$, $N = 2$ kdV hierarchy as a consistent reduction of the $N = 4$ KdV one. All the conserved quantities, as well as the above Lax formulations, are reduced to those of this $N = 2$ KdV hierarchy. However, there exists another consistent reduction of $N = 4$ KdV. Namely, one can choose the $SU(2)$ frame corresponding to the second or third options in (17) and also impose the conditions (25). Though before reductions these options are related to each other by the hidden $SU(2)$ symmetry, the reductions break $SU(2)$ down to $U(1)$ and so give rise to non-equivalent $N = 2$ KdV systems. One can show that under the second reduction all even-dimensional conserved $N = 4$ KdV hamiltonians vanish (their densities are proportional to Φ , or $\bar{\Phi}$) while the odd-dimensional ones go into those of the $a = -2$, $N = 2$ KdV hierarchy. For the flows corresponding to these hamiltonians this reduction is self-consistent in the sense that both the l.h.s and r.h.s. of the evolution equations for $\Phi, \bar{\Phi}$ vanish upon imposing (25). Thus two different $N = 2$ KdV hierarchy, the $a = 4$ and $a = -2$ ones, are encoded in the single $N = 4$ KdV hierarchy as its two non-equivalent reductions. The same property can be established in the $N = 1$ superfield formulation of $N = 4$ KdV system [18]. It would be interesting to find another Lax formulation of $N = 4$ KdV, such that the existence of these two reductions and the equivalence of different options in (17) were manifest. Hopefully, such a formulation can be constructed in harmonic superspace.

5. “Quasi” $N=4$ KdV hierarchy. In ref. [18] we have found one more integrable hierarchy with the small $N = 4$ SCA as the second hamiltonian structure. It was naturally assigned the name “quasi” $N = 4$ KdV hierarchy as the global $N = 4$ supersymmetry is explicitly broken down to $N = 2$ in this system. Also, it reveals no $SU(2)$ covariance and goes over to the $a = -2$, $N = 2$ KdV upon imposing the conditions (25). So it can be treated as an integrable extension of this $N = 2$ KdV hierarchy by chiral and anti-chiral superfields Φ , $\bar{\Phi}$.

It is interesting that, at cost of introducing new parameter c (not confuse it with the central charge!), the dimension 3 hamiltonian of this system (and actually all higher-order hamiltonians) can be written uniformly with the $a = 4, b = 0$ hamiltonian of genuine $N = 4$ KdV system

$$\mathcal{H}_3^c = \int dx d\theta d\bar{\theta} \left\{ J[D, \bar{D}]J - \frac{c-3}{3}J^3 - 4J\Phi\bar{\Phi} - 2c\Phi'\bar{\Phi} \right\}. \quad (26)$$

At $c = 1$ the hamiltonian (16) with $a = 4, b = 0$ is reproduced while at $c = 4$ one gets the quasi $N = 4$ KdV system which goes into the $a = -2$, $N = 2$ KdV hierarchy upon the reduction (25). Lacking $N = 4$ supersymmetry can be easily observed already at the level of linear pieces of the corresponding evolution equations

$$\dot{J} = -J''' + \dots, \quad \dot{\Phi} = -c\Phi''' + \dots, \quad \dot{\bar{\Phi}} = -c\bar{\Phi}''' + \dots. \quad (27)$$

Since $N = 4$ supersymmetry (15) linearly transforms J , Φ and $\bar{\Phi}$ through each other, $N = 4$ supercovariance strictly requires $c = 1$ in these equations. So, the case $c = 4$ clearly corresponds to the situation with broken $N = 4$ supersymmetry.

In [18, 17] a scalar Lax formulation for this hierarchy has been constructed

$$L = D \left(\partial + J - \Phi \partial^{-1} \bar{\Phi} \right) \bar{D}, \quad \frac{\partial L}{\partial t_k} = [L_{\geq 1}^{k/2}, L]. \quad (28)$$

Note that, like the $a = -2$, $N = 2$ KdV, this system admits also a matrix Lax formulation along the lines of ref. [21].

An interesting property of this new $N = 2$ hierarchy is that it gives rise, via consistent reductions, to two new lower-supersymmetry hierarchies with $N = 2$ SCA as the second hamiltonian structure. They were missed in the previous studies. One of them possesses only $N = 1$ global supersymmetry and no any kind of internal symmetry. The other possesses $U(1)$ symmetry but lacks supersymmetry. It is still different from the non-supersymmetric system constructed in [9]: it contains the mKdV hierarchy for the spin 1 current $j(t, x)$ in its bosonic sector, while in the system of ref. [9] this field satisfies the trivial equation, $\dot{j} = 0$. These observations suggest the existence of a “horizontal” sequence of hierarchies associated with the given SCA. It is parametrized by an extra parameter c which takes, similarly to the parameters a, b , some special values for the integrable cases. These systems range from the maximally supersymmetric one to lower-supersymmetric and even non-supersymmetric hierarchies. This conjecture implies that $N = 4$ SCA can serve as the second hamiltonian structure for more hierarchies, e.g., respecting only $N = 1$ supersymmetry or having no supersymmetry at all.

6. $N=4$ NLS-mKdV hierarchies. There exists a remarkable and well-known relation between (super)affine and (super)conformal algebras: the latter can be mapped on the former through various Sugawara-Feigin-Fuks (SFF) or coset constructions of (super)conformal stress-tensors in terms of the (super)affine currents. Being translated into the language of integrable

hierarchies, this correspondence manifests itself as the relation between two types of hierarchies: the KdV type ones associated with (super)conformal algebras as the second hamiltonian structure and the mKdV type ones which are hierarchies of evolution equations for the (super)affine currents with the (super)affine algebra as the hamiltonian structure. In this setting, the SFF representations for the stress-tensors come out as Miura-type maps between these two sorts of hierarchies.

Let us again apply to the KdV example. Introduce a spin 1 current $v(x)$ generating $\widehat{u(1)}$ affine algebra through the PB

$$\{v(x), v(y)\} = \partial_x \delta(x - y) . \quad (29)$$

Then, defining

$$u = v^2 + v' , \quad (30)$$

one observes that, as a consequence of PB (29), the so defined u generates a classical Virasoro algebra

$$\{u(x), u(y)\} = [-\partial^3 + 4u\partial + 2u']\delta(x - y) , \quad (31)$$

which is the same as in eq. (2). Thus eq. (30) gives the simplest example of SFF construction relating Virasoro algebra to the affine algebra $\widehat{u(1)}$. On the other hand, substituting (30) into the KdV hamiltonian in (2), one gets

$$\mathcal{H}_3 = \frac{1}{2} \int dx (v^4 + v'v') . \quad (32)$$

Through the PB (29) this hamiltonian gives rise to the evolution equation for v

$$\dot{v} = \{v, \mathcal{H}_3\} = -v''' - 6v'v^2 \quad (33)$$

which is the familiar mKdV equation. One can directly check that eq. (33) yields the standard KdV equation for u defined by eq. (30). Thus the SFF representation (30) is at the same time the Miura map relating KdV and mKdV hierarchies.

This correspondence more or less directly extends to the case of supersymmetric hierarchies. E.g., it is easy to check that an $N = 2$ superextension of the algebra $\widehat{u(1)}$

$$\{H(1), \bar{H}(2)\} = D\bar{D}\delta(1, 2), \{H(1), H(2)\} = 0, DH = \bar{D}\bar{H} = 0 , \quad (34)$$

H, \bar{H} being spin 1/2 fermionic chiral and anti-chiral superfields (actually, it collects two algebras $\widehat{u(1)}$ in its bosonic sector), yields just the $N = 2$ SCA (11) via the SFF construction

$$J = H\bar{H} + D\bar{H} + \bar{D}H . \quad (35)$$

After substituting this expression into the hamiltonian (10), one gets a set of evolution equations for H, \bar{H} which give rise to $N = 2$ mKdV hierarchies for the values of the parameter a listed in eq. (13). For J defined by eq. (35) one gets just the related $N = 2$ KdV hierarchies.

One can ask whether analogous underlying affine hierarchies can be found for $N = 4$ KdV and “quasi” KdV hierarchies. The answer is affirmative, though the proof is not straightforward.

First of all, it is clear that the relevant superaffine algebras should reveal some $N = 4$ structure. At present, explicit superfield constructions of superextensions of affine algebras and

superalgebras exist up to $N \geq 1$ (N is as before the number of independent $1D$ supercharges). In particular, $N = 2$ extensions exist for any affine (super)algebra admitting a complex structure [22]. It is natural to assume that a hidden $N = 4$ supersymmetry is inherent in those $N = 2$ affine (super)algebras which possess a quaternionic structure, namely those which contain as their local part the algebras listed in [23] (actually, this list can be readily extended to superalgebras). Then an $N = 2$ extension of two affine algebras $\widehat{u(1)}$ could be the simplest algebra of this sort. It contains in the bosonic sector just four copies of the $\widehat{u(1)}$ algebras, the set on which one can already define a quaternionic structure [23]. This $N = 2$ algebra is generated by two pairs of chiral and anti-chiral superfields H_α, \bar{H}_α , ($\alpha = 1, 2$) with the following PBs

$$\{H_\alpha(1), H_\beta(2)\} = 0, \quad \{H_\alpha(1), \bar{H}_\beta(2)\} = \delta_{\alpha\beta} D\bar{D}\delta(1, 2). \quad (36)$$

Indeed, it is easy to see the covariance of these relations, as well as of the chirality conditions for H_α, \bar{H}_α , under the transformations

$$\delta H_1 = \epsilon D\bar{H}_2, \quad \delta \bar{H}_1 = \bar{\epsilon} \bar{D}H_2, \quad \delta H_2 = -\epsilon D\bar{H}_1, \quad \delta \bar{H}_2 = -\bar{\epsilon} \bar{D}H_1. \quad (37)$$

They possess the same Lie bracket structure as (15) and so, together with the manifest $N = 2$ supersymmetry transformations, yield a representation of the same $N = 4$ supersymmetry. Hence, the above $N = 2$ affine algebra indeed supplies the simplest example of $N = 4$ affine algebra (its supercurrents form an $N = 4$ supermultiplet).

One may wonder whether it gives rise to “small” $N = 4$ SCA via some SFF construction and has any relation to $N = 4$ KdV hierarchy. However, it can be checked that one cannot construct, out of the superfields H_α, \bar{H}_α , any $N = 4$ multiplet of composite currents including $N = 2$ conformal stress-tensor with a Feigin-Fuks term (the latter is absolutely necessary for producing a central term in $N = 2$ SCA and thus generating at least an $N = 2$ KdV hierarchy as a subsystem). The only possibility is the $N = 4$ multiplet

$$\hat{J} = H_1\bar{H}_1 + H_2\bar{H}_2, \quad \hat{\Phi} = H_1H_2, \quad \hat{\bar{\Phi}} = \bar{H}_2\bar{H}_1, \quad (38)$$

which, via PBs (36), generates a topological (i.e. centreless) “small” $N = 4$ SCA. So, possible $N = 2$ affine hierarchies (even possessing rigid $N = 4$ supersymmetry) constructed on the basis of this $N = 2$ affine algebra seem to have no any direct relation to $N = 4$ KdV hierarchy.

Next in complexity is $N = 2$ extension of non-abelian affine algebra $\widehat{sl(2) \oplus u(1)}$ whose local bosonic part $sl(2) \oplus u(1)$ also contains four generators and is among the algebras given in [23]. The PBs of these $N = 2$ superalgebra read [22]

$$\{H(1), \bar{H}(2)\} = D\bar{D}\delta(1, 2) \quad (39)$$

$$\begin{aligned} \{H(1), F(2)\} &= DF\delta(1, 2), \quad \{H(1), \bar{F}(2)\} = -D\bar{F}\delta(1, 2), \\ \{\bar{H}(1), F(2)\} &= -\bar{D}F\delta(1, 2), \quad \{\bar{H}(1), \bar{F}(2)\} = \bar{D}\bar{F}\delta(1, 2), \end{aligned} \quad (40)$$

$$\{F(1), \bar{F}(2)\} = [(D + H)(\bar{D} + \bar{H}) + F\bar{F}]\delta(1, 2), \quad (41)$$

all other PBs vanishing. Here, H and \bar{H} satisfy the standard chirality conditions while F and \bar{F} are subject to the *nonlinear* version of chirality

$$(D + H)F = 0, \quad (\bar{D} - \bar{H})\bar{F} = 0. \quad (42)$$

These constraints are necessary for closure of Jacobi identities of the algebra [22].

Once again, it is a matter of direct calculation to check that these PBs together with the above linear and nonlinear chirality conditions are covariant under the following hidden *nonlinear* $N = 2$ transformations [19]

$$\begin{aligned}\delta H &= \epsilon D\bar{F} + \bar{\epsilon} H F, & \delta \bar{H} &= \bar{\epsilon} \bar{D} F - \epsilon \bar{H} \bar{F} \\ \delta F &= -\epsilon D\bar{H} - \epsilon(H\bar{H} + F\bar{F}), & \delta \bar{F} &= -\bar{\epsilon} \bar{D} H - \bar{\epsilon}(H\bar{H} + F\bar{F}).\end{aligned}\quad (43)$$

It can be easily checked that the above transformations, despite their non-linearity, indeed realize the same extra $N = 2$ supersymmetry as the transformations (15). Thus, combined with manifest $N = 2$ supersymmetry, they again form the previously defined $N = 4$ supersymmetry. The two pairs of affine $N = 2$ supercurrents F, \bar{F} and H, \bar{H} are unified into an irreducible $N = 4$ supermultiplet, so the $N = 2$ extension of $sl(2) \oplus u(1)$ algebra is in fact an $N = 4$ extension. This $N = 4$ structure might be made manifest by passing to $N = 4$ superfields, but we will not elaborate on this possibility here.

What is indeed important for our consideration is that this superaffine algebra allows for a transparent SFF construction of small $N = 4$ SCA on its basis. The explicit formulas expressing $N = 4$ supercurrents in terms of the affine supercurrents are as follows [19]

$$J = H\bar{H} + F\bar{F} + D\bar{H} + \bar{D}H, \quad \Phi = D\bar{F}, \quad \bar{\Phi} = \bar{D}F, \quad (D\Phi = \bar{D}\bar{\Phi} = 0). \quad (44)$$

These objects obey just the $N = 4$ SCA PB relations (11), (14) as a consequence of the affine PB relations (39) - (41). Due to the Feigin-Fuks term in J in (44) the resulting $N = 4$ SCA possesses a non-zero central charge, which, as was already mentioned, is crucial for getting $N = 4$ KdV system.

Now it is a standard routine to substitute these SFF expressions into the hamiltonians of $N = 4$ KdV hierarchy and to derive the flow equations for the affine currents H, \bar{H}, F, \bar{F} using the PBs of $N = 2$ affine $sl(2) \oplus u(1)$ algebra. Note that the lowest flow equations were derived in [19] also in another way, by the direct construction of the proper dimension $N = 4$ invariant hamiltonians and requiring them to be in involution. We do not present here these equations in view of their considerable complexity (see ref. [19]). We only notice the existence of a few consistent reductions of them.

The first one is effected by putting

$$F = \bar{F} = 0 \rightarrow \Phi = \bar{\Phi} = 0, \quad J = H\bar{H} + D\bar{H} + \bar{D}H. \quad (45)$$

This yields the $a = 4, N = 2$ mKdV ².

The second reduction goes as

$$H = \bar{H} = 0 \rightarrow J = F\bar{F}, \quad DF = \bar{D}\bar{F} = 0. \quad (46)$$

The resulting system is the $N = 2$ NLS hierarchy of refs. [24]. The existence of such a reduction has been firstly noticed in [15] at the level of $N = 4$ KdV hierarchy, with F and \bar{F} interpreted as the prepotentials of the spin 1 chiral supercurrents Φ and $\bar{\Phi}$ in a fixed gauge with respect to the prepotential gauge freedom.

²Using another choice of the frame with respect to the hidden automorphism $SU(2)$, it is possible to perform a reduction to the $a = -2, N = 2$ mKdV hierarchy as well.

This consideration shows that the $N = 4$ supersymmetric system constructed can be treated as $N = 4$ extension of at once two $N = 2$ supersymmetric hierarchies, $N = 2$ mKdV and NLS ones. This is why it has been named in [19] the “ $N = 4$ NLS-mKdV hierarchy”.

Needless to say, the “quasi” $N = 4$ KdV hierarchy can also be associated with some underlying $N = 2$ $\widehat{sl(2) \oplus u(1)}$ affine hierarchy. One simply substitutes the STT expressions (44) for the $N = 4$ supercurrents into the hamiltonians of the “quasi” $N = 4$ KdV hierarchy and Lax operator and derives the appropriate evolution equations for the affine supercurrents via the PB structure (39) - (41).

7. Concluding remarks. Thus, now we are aware of general method of setting up $N = 4$ supersymmetric KdV type hierarchies: each $N = 2$ affine algebra or superalgebra admitting a hidden $N = 4$ supersymmetry (quaternionic structure) can be used to construct such hierarchies via appropriate superfield SFF maps. We hope to naturally come in this way to $N = 4$ extensions of W algebras. An example of $N = 2$ affine algebra with $N = 4$ structure, next in complexity to $N = 2$ $\widehat{sl(2) \oplus u(1)}$, is the algebra $N = 2$ $\widehat{sl(3)}$. This case is under study.

There remain many conceptual and technical problems to be solved. In particular, it would be useful to work out convenient $N = 4$ superfield techniques of treating $N = 4$ hierarchies, based, e.g., on the harmonic superspace approach. Up to now, it has been successfully applied only to one example of $N = 4$ KdV type hierarchies, the $N = 4$ KdV system itself [13, 14]. An interesting unsolved problem is to construct super KdV hierarchy associated with the “large” $N = 4$ SCA as the second hamiltonian structure. There are indications that such a system could yield all the three known $N = 2$ supersymmetric KdV hierarchies as its different consistent reductions. One may also think about higher N hierarchies, say, with $N = 8$ supersymmetry. To my knowledge, no any “no-go” theorems are known which could forbid the existence of such systems. Also, it could happen that a number of the already known $N = 2$ hierarchies exhibit hidden higher N supersymmetries. For instance, in a recent preprint [25] it was found that $N = 4$ KdV hierarchy allows a map on the so called $(1, 1)$ GNLS (Generalized NLS) system [26] which involves one chiral fermionic and one chiral bosonic superfields.

Perhaps, the most urgent problem is to identify the place and to reveal possible implications of this novel wide class of supersymmetric integrable systems in the modern superstring and p -brane stuff. I believe this certainly can be done.

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